MATHEMATICAL ANALYSIS OF THE PARALLEL PARKING PROBLEM

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Almost every driver has had the experience of finding his automobile wedged between two other parked cars. A good driver can usually extricate his vehicle by performing an appropriate sequence of operations in an intuitive manner. The admissible operations include moving his car back

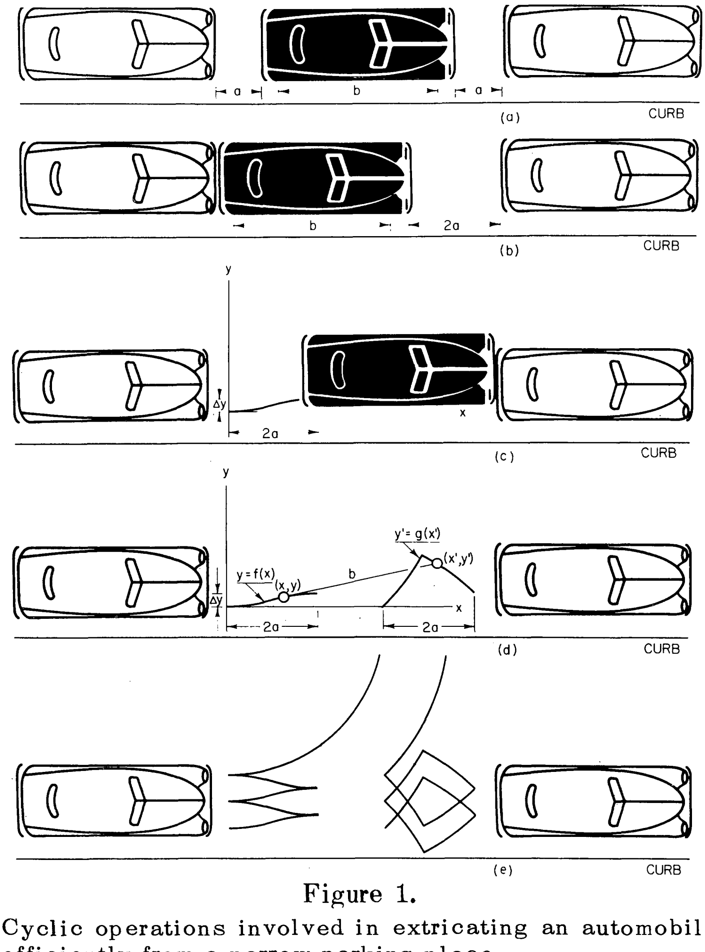

Figure 1.

Cyclic operations involved in extricating an automobile efficiently from a narrow parking place.

and forth, making virtual contact with the other automobiles, and concurrently manipulating the steering wheel. Consider Fig. 1a. The car in the middle has a wheelbase b, and is spaced the small distance a, both from the car in front and the car in the rear. Assume that the driver backs his car until his rear bumper virtually touches the front bumper of the car behind him. Figure 1b is assumed to be the initial configuration of the parallel parking problem which can be formulated as follows: *What cyclic operations must the driver perform in order to displace his automobile laterally sufficiently far to escape confinement from a narrow parallel parking place?*

If the confinement is close, the trapped automobile can only oscillate back and forth, and any lateral motion of the car will be small for each cycle. Since the car initially is parallel to the curb, the imposition of periodicity implies that the car becomes parallel to the curb again at the end of each cycle of operation. Consider a cartesian coordinate system with the origin determined by the initial position of the right rear wheel as shown in Fig. 1c, and let the x axis be directed parallel to the curb. Assume, Fig. 1d, that the initial cycle of operation produces a track of the right rear wheel specified by the relation

(1) $$y = f(x) ,$$

with the boundary conditions

(2) $$y(0) = 0 , \quad y(2a) = \Delta y ,$$
$$\dot{y}(0) \equiv \dot{y}(2a) = 0 ,$$

where the dots imply differentiation with respect to x. During the initial cycle of operation, the car is displaced the distance Δy laterally. It is sufficient to consider only the initial cycle since the second, or reverse cycle of operation, will be a mirror image of the first.

The track of the right rear wheel determines the track of the right front wheel. Erect a tangent at point (x, y) on the curve $y = f(x)$ and construct on the tangent the distance b to locate the point (x', y'); the locus of all such points is assumed to specify the track, $y' = g(x')$, of the right front wheel. The track of the right front wheel can be written in parametric notation

(3) $$x' = x + b(1+\dot{y}^2)^{-1/2} ,$$

(4) $$y' = y + b\dot{y}(1+\dot{y}^2)^{-1/2} .$$

Consider the integral

(5) $$W = \int y'dx' .$$

Substitute Eqs. (3) and (4) into Eq. (5) to obtain

(6) $$W = \int y dx - \int \frac{by\dot{y}d\dot{y}}{(1+\dot{y}^2)^{3/2}} + \int \frac{b\dot{y}dx}{(1+\dot{y}^2)^{1/2}} - \int \frac{b^2\dot{y}^2d\dot{y}}{(1+\dot{y}^2)^2} .$$

Integrate the second term on the right-hand side of Eq. (6) by parts to obtain

(7) $$-\frac{b}{2}\int \frac{2y\dot{y}\,d\dot{y}}{(1+\dot{y}^2)^{3/2}} = -\frac{b}{2}\left[-\frac{2y}{(1+\dot{y}^2)^{1/2}} + \int \frac{2\dot{y}\,dx}{(1+\dot{y}^2)^{1/2}}\right] .$$

Substitute Eq. (7) into Eq. (6) to obtain

(8) $$\int y'dx' - \int y\,dx = \frac{by}{(1+\dot{y}^2)^{1/2}} - b^2\int \frac{\dot{y}^2 d\dot{y}}{(1+\dot{y}^2)^2} .$$

Integrate Eq. (8) over a complete cycle. Imposition of the boundary conditions, Eqs. (2), reduces Eq. (8) to

(9) $$\int y'dx' - \int y\,dx = b\Delta y .$$

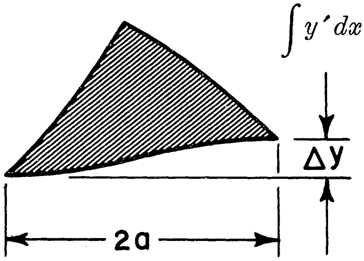

Figure 2
Area enclosed by tracks of right front and rear wheels. The lateral distance Δy is a maximum whenever the enclosed area is maximum.

Figure 2 is a sketch illustrating the physical interpretation of the integrals of Eq. (9). The top curve of Fig. 2 represents the track of the right front wheel, the bottom curve represents the track of the right rear wheel. The area enclosed by the two curves, from Eq. (9), is a direct measure of the distance Δy that the car is displaced laterally in one cycle. In order to maximize the distance Δy that the car moves laterally for each cycle of operation, the area shown in Fig. 2 must be a maximum.

Consider the trajectory of the right rear wheel that results in maximum displacement Δy. The lengths r_1 and r_2 in Fig. 3 are the respective minimum left turning radius and right turning radius of the right rear wheel. If

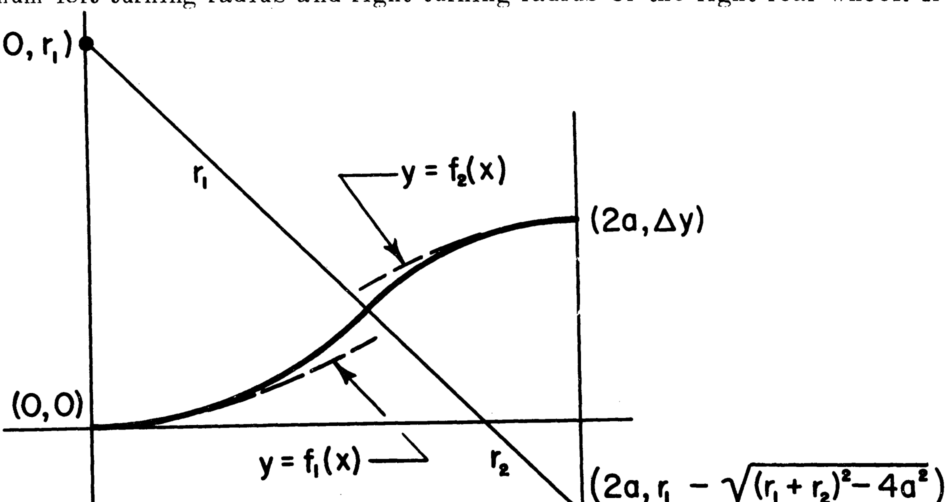

Figure 3
Trajectory of the right rear wheel treated as arcs of a circle.

t is the *tread* of the car, then

(10) $$r_1 = r_2 + t \; .$$

The dashed lines in Fig. 3 represent arbitrary curves $y_1(x)$ and $y_2(x)$ with the respective bounded curvatures $1/\rho_1 \leqq 1/r_1$ and $1/\rho_2 \leqq 1/r_2$. Each of the dashed lines lies either on or outside of the circle determined by its associated arc. This fact is inferred by considering the arcs as the limits of the arbitrary curves as their respective curvatures approach $1/r_1$ and $1/r_2$. As $1/\rho_1 \to 1/r_1$, for example, the points on $y_1(x)$ move counterclockwise with reference to the origin and toward the circular arc. Thus, Δy is maximum when $y_1(x)$ and $y_2(x)$ are circular arcs. In this case

(11) $$\Delta y = (r_1 + r_2) - [(r_1 + r_2)^2 - 4a^2]^{1/2} \; .$$

For the special case where Δy is maximum the area of Fig. 2 is determined by circular arcs of radii r_1, r_2, $(r_1^2 + b^2)^{1/2}$, and $(r_2^2 + b^2)^{1/2}$. Each wheel turns through the same arc; that is, $\sin^{-1} 2a/(r_1 + r_2)$. At any instant of time all four wheels are rotating around a common point. It can be verified by integration, or otherwise, that the area of Fig. 2 is given by $b\Delta y$ where Δy is specified by Eq. (11).

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